ON DECIDABLE, FINITELY AXIOMATIZABLE, MODAL AND TENSE LOGICS WITHOUT THE FINITE MODEL PROPERTY

PART I

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ABSTRACT

Decidability results in modal and tense logics were obtained through the finite model property. This paper shows that the method is limited, since there exists a decidable extension of modal T that lacks the finite model property. The decidability of the system is proved through a new method, the *reduction method*, (using a theorem of Rabin).

Introduction

We give an example of the incompleteness of the technique of using the finite model property (f.m.p.) to prove decidability in modal logic. We present a decidable, finitely axiomatizable normal logic between T and $S4$ which lacks the f.m.p. Our strategy is the following:

We shall define an auxiliary tense system M_* in § 1. In § 2 we shall provide it with semantics. In § 3 we shall show (by methods of [3]) that it lacks the finite model property. In § 4 we show that M_* is decidable. Finally in 5 §we shall construct a modal system G_* which is a finitely axiomatizable extension of T which lacks the fiinite model property. We shall also give a 1-1 interpretation of G_* in M_*, and thus obtain the decidability of G_*.

The particular systems M_* and G_* are of no special importance (except for the fact that G_* is between T and $S4$). The method of proof however, especially the proof of decidability, is of great interest as it presents a new technique for obtaining decidability results in modal logics.

1. The system M_*

Our language contains, besides the connectives of classical propositional logic,

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the connectives $G\phi$ (ϕ will always be true) and $Y\phi$ (ϕ was true yesterday). Our axioms are:
all the truth functional tautologies, modus ponens, substitution and:

AXIOMS.*

(1) $G(\phi - \psi) \to (G\phi \to G\psi)$

(2) $Y(\phi \to \psi) \to (Y\phi \to Y\psi)$

(3) $G\phi \to GG\phi$

(4) $\phi \to \sim G \sim Y\phi$

(5) $\sim G(G \wedge \sim\phi)$; $\sim Y(\phi \wedge \sim\phi)$

(6) $Y \sim \phi \leftrightarrow \sim Y\phi$

(7) $\sim\phi \to Y \sim G\phi$

(8) $\sim G \sim Y \sim \phi \to \sim\phi \vee \sim G\phi$

(9) $\vdash \phi \Rightarrow \vdash G\phi, \vdash Y\phi$. (this is a group of inference rules).

2. Semantics

Let T be any nonempty set (thought of as a set of moments of time), T-structures are of the form $\underline{A} = (A_t, <, r, 0)$ $t \in T$ where A_t is a function assigning values 0, 1 to each atomic sentence. $<$ is a binary relation on T, r is a unary function on T, $0 \in T$ and the following holds:

(10) $<$ is transitive

(11) $r(x) < x$, $x \in T$

(12) $\forall x \exists y (x < y)$

(13) $\forall x \exists y (x = r(y))$.

(14) $x < y \to (x < r(y)$ *or* $x = r(y))$.

Satisfaction is defined as follows:

Let A be a T-structure and let $x \in T$, then the value of ϕ at x (notation $[\phi]_x$) is defined by induction:

(15) $[p]_x =$ value given by A_x, for atomic p.

(16) $[\phi \wedge \psi]_x = 1$ iff $[\phi]_x = 1$ and $[\psi]_x = 1$
$[\sim\phi]_x = 1$ iff $[\phi]$ $[\phi]_x = 0$

(17) $[G\phi]_x = 1$ iff $[\phi]_y = 1$ for all y such that $x < y$

(18) $[Y\phi]_x = 1$ iff $[\phi]_{r(x)} = 1$.

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* Our axioms and conditions are not independent. We do not bother with independence in this paper, as all we want is to show the existence of a certain kind of modal logic.

We say that a sentence ϕ is valid iff for all nonempty T and all T-structures $[\phi]_0 = 1$.

THEOREM 19. *All axioms are valid.*

PROOF.

(3) follows from (10).

(4), (5) follows from (11), (12) and (13).

(6) follows from r being a function

(7) follows from (12).

(8) follows from (14).

The reader can verify that the inference rules preserve validity.

DEFINITION. A theory Δ is complete iff for all ϕ, $\phi \in \Delta$ or $\sim\phi \in \Delta$.

LEMMA 20. *Let Δ be a complete and consistent theory then:*

(a) *If $\sim G\phi \in \Delta$ then there exists a complete theory Δ^{ϕ} such that $\sim\phi \in \Delta^{\phi}$ and for every ψ such that $G\psi \in \Delta$ be have that $\psi \in \Delta^{\phi}$.*

(b) *If $\sim Y\phi \in \Delta$ then a Δ_{ϕ} exists with similar properties.*

(c) *There exists a complete and consistent Δ^s such that*

(c1) $G\psi \in \Delta \rightarrow \psi \in \Delta^s$

(c2) $\psi \in \Delta \rightarrow Y\psi \in \Delta^s$.

PROOF. The way to get (a) and (b) is well known and is due to Makinson and Scott. To get (c) we follow Lemmon-Scott:

Assume $\{\psi \mid G\psi \in \Delta\} \cup \{Y\phi \mid \phi \in \Delta\}$ is not consistent, so for some ψ_i, ϕ_j we have:

$$\vdash \psi_1 \wedge \cdots \wedge \psi_n \rightarrow \sim(Y\phi_1 \wedge \cdots \wedge Y\phi_m)$$
$$\vdash \psi_1 \wedge \cdots \wedge \psi_n \rightarrow \sim Y(\phi_1 \wedge \cdots \wedge \phi_m)$$
$$\vdash G(\psi_1 \wedge \cdots \wedge \psi_n) \rightarrow G \sim Y(\phi_1 \wedge \cdots \wedge \phi_m)$$
$$\vdash G\psi_1 \wedge \cdots \wedge G\psi_n \rightarrow G \sim Y(\phi_1 \wedge \cdots \wedge \phi_m)$$
$$\vdash \sim G \sim Y(\phi_1 \wedge \cdots \wedge \phi_m) \rightarrow \sim (G\psi_1 \wedge \cdots \wedge G\psi_n)$$

but $\vdash \phi_1 \wedge \cdots \wedge \phi_m \rightarrow \sim G \sim Y(\phi_1 \wedge \cdots \wedge \phi_m)$. So $\sim (G\psi_1 \wedge \cdots \wedge G\psi_n) \in \Delta$ but $\wedge \phi_i \in \Delta$, a contradiction.

Now extend this set to a complete theory Δ^s.

DEFNITION 21. $\Delta < \Delta'$ iff for every ψ, $G\psi \in \Delta \rightarrow \psi \in \Delta'$.*

LEMMA 22. *$\Delta < \Delta'$ and $\Delta' < \Delta''$ implies $\Delta < \Delta''$.*

* From now on $\Delta, \Theta, \ldots$ range over complete and consistent theories.

PROOF. Use $\vdash G\phi \to GG\phi$.

LEMMA 23. *For any* ϕ, ψ: *If* $\sim Y\phi \in \Delta$ and $\sim Y\psi \in \Delta$ *then* $\Delta_\phi = \Delta_\psi$.

PROOF. Let $\alpha \in \Delta_\phi$, $\sim \alpha \in \Delta_\psi$.
We know that for all β, $Y\beta \in \Delta \to \beta \in \Delta_\phi$ and $\beta \in \Delta_\psi$. Therefore $\sim Y\alpha \in \Delta$ and so since $\vdash \sim Y\alpha \leftrightarrow Y \sim \alpha$ we get that $Y \sim \alpha \in \Delta$ and so $\sim \alpha \in \Delta_\phi$. A contradiction.
From now on let us call the unique theory of (23), Δ_Y.

LEMMA 24. *If* $\sim Y\phi \in \Delta$ *then* $\Delta_\phi < \Delta$.

PROOF. Let $G\psi \in \Delta_\phi$, and $\sim\psi \in \Delta$. Since $\vdash \sim\psi \to Y \sim G\psi$ we get that $\sim G\psi \in \Delta_\phi$, a contradiction.

LEMMA 25. *If* $\sim Y\phi \in \Delta$ *and* Δ^s *is as in Lemma* (20c) *then* $(\Delta^s)_\phi = \Delta$.

PROOF. Let $\sim \alpha \in \Delta$, and $\alpha \in (\Delta^s)_\phi$. Now $\sim\alpha \in \Delta$ implies $Y \sim \alpha \in \Delta^s$ by construction. But $\vdash Y \sim \alpha \leftrightarrow \sim Y\alpha$ and so $\sim Y\alpha \in \Delta^s$ and so $\sim \alpha \in (\Delta^s)_\phi$ by (23), a contradiction.

LEMMA 26. $\Theta < \Delta \to (\Theta < \Delta_\phi$ *or* $\Theta = \Delta_\phi)$.

PROOF. Let $G\psi \in \Theta$ and $\sim\psi \in \Delta_\phi$ and $\alpha \in \Theta$ and $\sim\alpha \in \Delta_\phi$.
$\vdash \beta \wedge G\beta \to G \sim Y \sim \beta$ (8).

Since $(\alpha \vee \psi) \wedge G(\alpha \vee \psi) \in \Theta$ we have that $\sim Y \sim (\alpha \vee \psi) \in \Delta$ and so $Y(\alpha \vee \Psi) \in \Delta$ and so $\alpha \vee \psi \in \Delta_\phi$, a contradiction.

Let Θ be a given complete theory. We now turn to construct a model $\underline{A}$ of Θ. One simple way of doing this is to take T as the set of all consistent and complete theories, to define $<$ on T, to define r, etc. and get the model in the usual manner using Lemmas 20–26. Since we are interested in proving that M_* is decidable we must construct the model more carefully. We begin with some definitions.

Let N^* be the set of all finite sequences of natural numbers and let the empty sequence $\Lambda \in N^*$. Let * denote concatenation of sequences, and define $x \in N^*$, the following functions called successorship functions:

(a) $y = r_0(x) = x^*(0)$.

(b) $y = p_n(x) = x^*(2n + 1)$, $n \geqq 0$.

(c) $y = s_n(x) = x^*(2n + 2)$, $n \geqq 0$.

y is said to be a successor of x.

For each complete and consistent theory Δ and each $\sim G\phi \in \Delta$, let Δ^ϕ be one fixed theory fulfilling (20a). Similarly choose for each Δ a fixed Δ^s as in (20c). From now on Δ^ϕ and Δ^s become unique whenever defined.

Let Θ be a given complete and consistent theory. Let $\underline{\underline{M}}$ be the smallest family of theories such that $\theta \in \underline{\underline{M}}$ and whenever $\Delta \in \underline{\underline{M}}$ we have that Δ_y, $\Delta^s \in \underline{\underline{M}}$ and also $\Delta^{\phi} \in \underline{\underline{M}}$ for any ϕ such that $\sim G\phi \in \Delta$. $\underline{\underline{M}}$ is denumerable; so let $\Gamma_1, \Gamma_2, \cdots$ be an enumeration of the elements of $\underline{\underline{M}}$.

We now proceed to define a subset $T \subseteq N^*$ and a function Δ: $T \to \underline{\underline{M}}$ as follows

Stage 0. Let $\Lambda \in T$ and let $\Delta(\Lambda) = \Theta$.

Stage 1.

a) Let $x = r_0(\Lambda) \in T$ and let $\Delta(x) = \Delta(\Lambda)_y$

b) If $\Gamma_i \in \underline{\underline{M}}$, is such that $(\Gamma_i)y = \Delta(\Lambda)$,

then let $x = s_i(\Lambda) \in T$ and let $\Delta(x) = \Gamma_i$.

c) Let Γ_n be such that

1) Γ_n is of the form $\Delta(\Lambda)^{\phi}$ for some ϕ.

2) For no k do we have $\Delta(\Lambda) = (\Gamma\ )y \cdots y$ ($k$times)

then let $x = p_n(\Lambda) \in T$ and let $\Delta(x) = \Gamma_n$.

Stage $n+2$. Let x be a point in T that was put in T at stage $n+1$. We distinguish two cases. Let x be a successor of a $y \in T$.

1) If x is an r_0 or p_n successor, repeat the construction as in stage 1 using $\Delta(x)$ as if it were Θ and define the successorship relation like in the case of Θ of stage 1.

2) In case x is an s_n successor, do as in previous cases except *do not follow* instruction (a) of stage 1. Thus in this case $r_0(x)$ is not in T.

DEFINITION 27. Let $\leqq_1$ be the relation on T of being an s, or p successor (i.e. no use is made of r_0). Let $\leqq$ be the transitive closure of $\leqq_1$.

Let $y > x$ be the transitive closure of the relation of being an r_0 successor (i.e $x = r_0^m(y)$ for some m), (we write $>$ since r_0 is in the past of y and so $y > x$ means x is the past of y, $x \leqq y$ means the same).

LEMMA 28. $x \leqq y \to \Delta(x) < \Delta(y)$.

PROOF. By construction and (22), and definition of Δ^s and of s successorship.

LEMMA 29. $x > y \to \Delta(y) < \Delta(x)$.

PROOF. By construction and by (22) and (24).

DEFINITION 30. (a) $r(x) = r_0(x)$ if $r_0(x)$ exists (note that in case $x = s_n(y)$ for some n we do not construct $r_0(x)$), and $r(x) = y$ if $x = s_n(y)$ for some n.

(b) Define $x < y$ as follows:

(b1) If there exists a finite set of points $y_0, y_1 \cdots y$ such that $y_0 > x$ and $y_0 \leqq y_1$ and $y_1 > y_2$ and $y_2 \leqq y_3$ and $y_3 > y_4$ and $y_4 \leqq y_5 \cdots$ until y.

(b2) If there exists a finite set of points $y_0, y_1 \cdots y$ such that $x \leqq y_0$ and $y_0 > y_1$, $y_1 \leqq y_2$, $y_2 > y_3$, $y_3 \leqq y_4 \cdots$ until y.

THEOREM 31. (a) $\Delta(r(x)) < \Delta(x)$. (b) $x < y \rightarrow \Delta(x) < \Delta(y)$.

PROOF.* (a) follows by (24) and (25).

The proof of (b) is by induction on the number of points $y_0 \cdots y_n$.

Case $n = 0$. In this case $y_0 = y$ and follows from (28) and (29).

Case $n + 1$. We distinguish two subcases:

(I) $y_n > y = y_{n+1}$, we assume $y = r_0^m(y_n)$, $m \geqq 1$. In this case we know by the induction hypothesis that $\Delta(x) < \Delta(y_n)$. We want to show that $\Delta(x) < \Delta(y)$.

Now since $n \geqq 0$ we have that $y_{n-1} \leqq y_n$ (y_{n-1} may be taken as x in case $n = 0$).

Let y' be the $\leqq$ *predecessor of* y_n so $\Delta(y') < \Delta(y_n)$ by construction. Since $y_n > y$ we have that $r_0(y_n)$ exists and that $(\Delta(y_n))_\psi \neq \Delta(y')$. So by (26) $\Delta(y') < (\Delta(y_n))_\psi = \Delta(r_0(y_n))$. Assume by induction that $\Delta(y') < \Delta(r_0^i(y_n))$ and show that $\Delta(y') < \Delta(r_0^{i+1}(y_n))$. By construction $\Delta(y') \neq \Delta(r_0^{i+1}(y_n))$ and therefore by (26) we have $\Delta(y') < \Delta(r_0^{i+1}(y_n))$, since $\Delta(r_0^{i+1}(y_n)) = \Delta(r_0^i(y_n))_y$. Thus $\Delta(y') < \Delta(y)$ and by (24) and by (29) and the induction hypothesis for y', we get $\Delta(x) < \Delta(y)$.

(II) In this case we assume that $y_n \leqq y$. By the induction hypothesis $\Delta(x) < \Delta(y_n)$ and by (28) and (22) $\Delta(x) < \Delta(y)$.

THEOREM 32. *Notice that in the definition of $x < y$ we used a set of points $y_0, y_1 \cdots y$ with a special property. If case* (b2) *of* (3) *held then $y_0 > y_1$. This means that y_1 is "higher" than y_0 in the tree* (*it is $r_0^m(y_0)$ successor*), *and each y_{n+1} is higher in the tree than y_n. In the case of* (b1) *again we get that y_1 is higher in the tree than y_0 and y_{n+1} is higher in the tree than y_n. So if we take $x\rho y$ to mean y is higher than x (through any kind of successorship), we get that $y_0 \rho y_1 \cdots y$ (i.e., they are linearly ordered).*

DEFINITION 33. We now construct the model $\underset{=}{A}$. Let T be our tree, let $<$, r be as in definition (30). Define $[p]_x = 1$ iff $p \in \Delta(x)$, for a propositional variable p.

LEMMA 34. $\underset{=}{A}$ *is a well defined model.*

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* I am indebted to the reviewer for correcting the proof.

PROOF.

(10) holds since by definition $<$ is transitive.

(11) also holds by the definition of $r(x)$ (30).

(12a) holds since $\vdash \sim G(\phi \wedge \sim\phi)$ and (13) since Δ^s is always defined and $\in M$.

(14) holds; the proof is as follows:

Assume $x < y$ so for some $y_0 \cdots y_{n-1}$ the following holds:

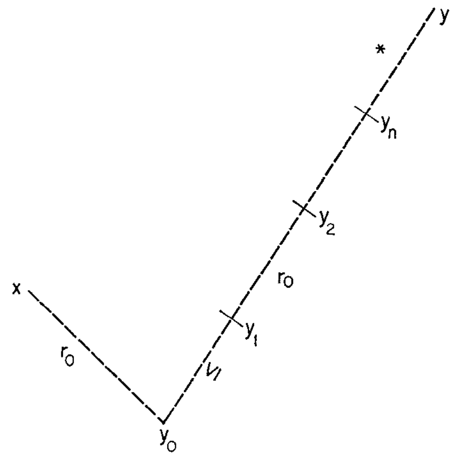


Fig. 1

$x = r_0^m(y_0)$ $m \geqq$ (i.e.. x may be y_0 which is b2 of (30)). $y_0 \leqq y_1, y_1 > y_2, y_2 \leqq y_3 \cdots$ until y.

Case $n \geqq 1$. In this case no matter whether $r(y) = r_0(y)$ or is the predecessor of y we have that $x < r(y)$.

Case $n = 0$. i.e., $y = y_1$.

Now if $x = r_0^m(y_m)$ with $m \neq 0$ then $x < r(y)$.

If $x = y_0$ and y is not a successor of x then again $x < r(y)$. In case $x = y_0$ and y is a successor of x then either $r(y) = r_0(y)$ in which case $x < r(y)$ by (26), or $r(y) = x$ which is (14).

LEMMA 35. $[\phi]_x = 1$ *iff* $\phi \in \Delta(x)$, *for all* ϕ.

PROOF. For propositional variable this holds by definition. $\Lambda, \sim$ present no difficulties.

Let $G\phi \in \Delta(x)$ and let $x < y$. Then by (31), $\Delta(x) < \Delta(y)$, i.e., $\phi \in \Delta(y)$ and so $[\phi]_y = 1$.

Let $\sim G\phi \in \Delta(x)$. So $\sim\phi \in (\Delta(x))^{\phi} = \Gamma$. If $\Delta(x) \neq \Gamma y \cdots y$ (k times) for all k, then by definition $\Gamma = \Delta(y)$ for $x \leqq_1 y$, otherwise $\Delta(x) = \Gamma y \cdots y$ (k times). Let $\Gamma = \Gamma n_1 \in M$ and $\Gamma y \cdots y$ (i times) be $\Gamma n_i \in M$ (M is closed under the $\Delta \mapsto \Delta y$ operation). Then by the definition of stage n, case (1b) we have that $y = s_{n_1}(s_{n_2}(\cdots s_{n_k}(x)\cdots) \in T$ and $\Delta(y) = \Gamma$ and $x \leqq y$.

Let $Y\phi \in \Delta(x)$ so $\phi \in (\Delta(x))_y$; regarding $r(x)$, if $r(x) = r_0(x)$ which is $(\Delta(x))_\psi$ we are finished and if $r(x)$ is the predecessor of x, then it is the case only because $(\Delta(x))_y$ equals the predecessor of x and so again we are finished.

3. M_* lacks f.m.p.

We use the results of Makinson [3]. Makinson extended T with $\Box(\Box^2\phi \to \Box^3\phi) \wedge \Box\phi \to \Box^2\phi$ and this system lacks f.m.p., since $\Box p_0 \to \Box\Box p_0$ is not provable and any model in which it is false is infinite.

Our use of this result is as follows: we interpret Makinson's system in M_* in such a manner as to have that all Makinson's axioms hold and $\Box p_0 \to \Box\Box p_0$ does not. Therefore there cannot be a finite model of M_* in which $\Box p_0 \to \Box\Box p_0$ is false since this gives rise to a finite model of Makinson's sytem in which $\Box p_0 \to \Box\Box p_0$ is false.

Definition 36. $\Box\phi = \operatorname{def} Y(\phi \wedge G\phi)$.

Theorem 37.

(a) $\vdash \Box\phi \to \phi$

(b) $\vdash \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$

(c) $\vdash \Box(\Box^2\phi \to \Box^3\phi) \wedge \Box\phi \to \Box^2\phi$

(d) $\vdash \phi \Rightarrow \vdash \Box\phi$

(e) $\not\vdash \Box\phi \to \Box\Box\phi$

Proof. We shall prove only (c) and (e).
By definition (36) we see that:

(38) $[\Box\phi]_x = 1$ iff $[\phi]_y = 1$ for all y such that $r(x) < y$ or $r(x) = y$.

Assume now that $[\Box\phi]_x = 1$ and $[\Box(\Box^2\phi \to \Box^3\phi)]_x = 1$ and $[\Box^2\phi]_x = 0$. By (13) $x = r(x_0)$ for some x_0. Consider Fig. 2.

$[\Box^2\phi]_x = 0$ so for some y such that $y = r(x)$ or $r(x) < y$ we have that $[\Box\phi]_y = 0$.

Now for some z such that $z = r(y)$ or $r(y) < z$ we have that $[\phi]_z = 0$. We distinguish some cases:

(a) $r(x) < y$:

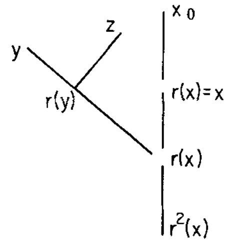


Fig. 2

By (14) $r(x) < r(y)$ or $r(x) = r(y)$.

(a1) $r(x) < r(y)$: Since $r(y) = z$ or $r(y) < z$ we get that $r(x) < z$ and so $[\phi]_z = 1$ since $[\Box\phi]_x = 1$. A contradiction.

(a2) $r(x) = r(y)$. Again if $r(y) < z$, a contradiction and if $r(y) = z$, again a contradiction.

(b) Therefore $r(x) = y$: So $r^2(x) = z$ or $r^2(x) < z$ holds. This yields that $[\Box^3\phi]_{x_0} = 0$. Since $r(x_0) = x < x_0$ we must have that $[\Box^2\phi \to \Box^3\phi]_{x_0} = 1$ and therefore $[\Box^2\phi]_{x_0} = 0$. So at some u such that $u = x$ or $x < u[\Box\phi]_u = 0$. u cannot be x and so $x < u$. Therefore for some v such that $v = r(u)$ or $r(u) < v$ we have that $[\phi]_v = 0$.

(c) Case $v = r(u)$: Since $x < u$, by (14) we conclude that $x = r(u)$ or $x < r(u)$ and since $r(x) < x$ we get that $r(x) < r(u)$ and so $r(x) < v$.

(d) $r(u) < v$: In this case we also get that $r(x) < v$. Now we have a contradiction since $[\Box\phi]_x = 1$.

To show that $\Box p \to \Box\Box p$ is not a theorem let $[p]_{r(x)} = 1$ and $[p]_y = 1$ for all y such that $r(x) < y$. Let $[p]_{r^2(x)} = 0$ so $[\Box p \to \Box\Box p]_x = 0$.

4. Decidability of M_*

Let N^* be the set of all finite sequences of natural numbers. Let $*$ denote concatenation of sequence. Define p_n, s_n, r_0 as before, i.e., $p_n(x) = x * (2n+1)$, $s_n(x) = x * (2n+2)$ and $r_0(x) = x * (0)$. Thus, p_n, s_n, r_0, $n \geqq 0$ are functions: $N^* \nrightarrow N^*$. Let Λ denote the empty sequence. We shall refer to p_n, s_n, r_0 as successor functions. Let $x \leqq_1 y$ mean $y = p_n(x)$ for some $n, x \leqq_2 y$ mean $y = s_n(x)$ for some n and $x \rho y$ mean $x \leqq_1 y \vee x \leqq_2 y$. Let $\leqq$ be the transitive closure of ρ. Let R be the transitive closure of $x\rho y \vee y = r_0(x)$.

THEOREM 40. (*Rabin* [4]). *The monadic second-order theory (with variables for finite sets also) of* $(N^*, \Lambda, s_n\, p_m, r_0, \leqq, R)$ *is decidable.*

In order to get that M_* is decidable we have to express its semantics in this language. We gave the completeness theorem in §2 with anticipation for this section.

DESCRIPTION OF SEMANTICS

(41) Let $x > y$ mean $\exists m(y = r_0^m(x))$. We can define this in terms of the other relations.

(42) Let $T_0 \subseteq N^*$ be such that the following holds:

(a) $x \in T_0 \rightarrow s_0(x) \in T_0$

(b) $0 \in T_0$

(c) if $y \notin T_0$ then no successor of y is in T_0

(d) if $x \in T_0 \wedge x \leqq_2 y$ then $r_0(y) \notin T_0$

(e) if $x \in T_0$ and $x \leqq_1 y$ then $r_0(y) \in T_0$

(f) $r_0(\Lambda) \in T_0$. $\forall x \exists u(x \leqq u)$.

Define $r(x) = r_0(x)$ if $r_0(x) \in T_0$ and $r(x) = y =$ predecessor of x otherwise (i.e. in case $y \leqq {}_2 x$) for $x \in T_0$.

Define $x < y$ iff there exists a finite set C, linearly ordered by R, such that:

(1) If y_0 is the first element of C, then either $x = y_0$ or $x = r_0^m(y_0)$ for some m.

(2) For every $z_0, z_1, z_2 \in C$ such that z_1 is a successor in C of z_0 and z_2 is a successor in C of z_1 the following holds: if $z_0 \leqq z_1$ then $z_1 > z_2$ and if $z_0 > z_1$ then $z_1 \leqq z_2$.

(3) For every $z_0, z_1 \in C$ such that z_1 is successor of z_0 we have that $z_0 > z_1$ or $z_0 \leqq z_1$.

(4) $y_0 \leqq$ the successor of y_0 in C.

(5) $y \in C$.

REMARK 43. (42) (a)–(f) are all definable in our language. Similarly $r(x)$, $x < y$ and whatever is needed.

THEOREM 44. *M_* is complete for this semantics.*

PROOF. Our definitions here are exactly the definitions of §2 (for the definition of $x < y$ see Theorem 32). So Lemmas 43 and 35 give us completeness.

THEOREM 45. *M_* is primitively recursively decidable.*

PROOF. We express in the language of N^* the assertion that ϕ is true in all models for any formula ϕ. To do this we associate with any formula ϕ a formula $\phi^*(y)$ in the language of N^* with y free as follows:

$(p_i)^* = y \in C_i$, with p_i a propositional variable and C_i set variable.

$(\phi \wedge \psi)^* = \phi^* \wedge \psi^*$

$(\sim \phi)^* = \sim \phi^*$

$(G\phi)^* = y \in T_0 \wedge (\forall z \in T_0)(y < z \rightarrow \phi^*(z))$, where T_0 is a set variable and $<$ is the relation defined in (42)–(43) above.

$(Y\phi)^* = (z = r(y)) \wedge \phi^*(z)$ where r is the function defined in (42)–(43) above.

What $\phi^*(y)$ says is that ϕ is true at the point y. (The reader can verify this by induction, taking $C_\iota = \{x \mid p_i \text{ true at } x\}$). We shall now say that ϕ is true at all models, i.e., ($\forall T_0$ such that (42) holds) $(\forall C_1 \cdots C_n \subseteq T_0)(\forall y \in T_0)\phi^*(y)$, where ϕ has $p_i \cdots p_n$ as atomic propositions. Now since this sentence is decidable in N^*, M_* is decidable.

5. A decidable, finitely axiomatizable extension of T without the finite model property

We have seen in the preceding sections that M_* does not have the f.m.p. and is decidable. We have interpreted Makinson's system in M_* in such a manner as to have Theorem 37.

In this section we show that G_*, the extension of Makinson's system, obtained by taking all modal sentences holding in M_*, is finitely axiomatizable.

G_* is the extension of Makinson's system with the following axioms:

(52) $$\phi \wedge \Box\psi \wedge \sim \Box^2\psi \rightarrow \Diamond(\Diamond\phi \wedge \sim\Box\psi)$$

(53) $$\Box\alpha \wedge \phi \wedge \Box\psi \wedge \sim\Box^2\psi \rightarrow \Diamond(\Diamond\phi \wedge \Box^2\psi \wedge \sim\Box^3\psi \wedge \Box^2\alpha)$$

(54) $$\Box\psi \wedge \sim\Box^2\psi \rightarrow (\Box(\sim\Box\psi \rightarrow \alpha) \vee \Box(\sim\Box\psi \rightarrow \sim\alpha))$$

(55) $$\Box\psi \wedge \sim\Box^2\psi \wedge \Diamond(\Box\alpha \wedge \sim \Box\psi) \rightarrow \Box^2\alpha$$

(56) $$\Box\phi \wedge \Box\psi \wedge \sim\Box^2\psi \rightarrow \Box(\Box\psi \rightarrow \Box\phi).$$

THEOREM 60. *Taking $\Box\phi$ as $Y(\phi \wedge G\phi)$ we get that all axioms of G_* are valid in M_*.*

LEMMA 61. *Let $[\Box\phi]_x = 1, [\sim\Box^2\phi]_x = 1$ then $r(x)$ is the only point y such that $[\sim\Box\phi]_y = 1$ and $y = r(x)$ or $r(x) < y$ hold.*

PROOF. See Theorem 37, Cases (a) and (b).

Let us now examine each axiom.

AXIOM 52. If $[\phi \wedge \Box\psi \wedge \sim\Box^2\psi]_x = 1$ then $[\sim\Box\psi \wedge \Diamond\phi]_{r(x)} = 1$.

AXIOM 53. If the antecedent is true at x then the consequent is true at x_0 where $r(x_0) = x$.

AXIOM 54. This follows from (38) and from (61).

AXIOM 55. Let $[\Box x \wedge \sim \Box^2\psi]_x = 1$ and let $[\Diamond(\Box\alpha \wedge \sim \Box\psi]_x = 1$, so for some y such that $r(x) = y$ or $r(x) < y$ $[\Box\alpha \wedge \sim \Box\psi]_y = 1$, by (61) $y = r(x)$. Therefore $[\Box\alpha]_{r(x)} = 1$ and so for every z such that $r^2(x) = z$ or $r^2(x) < z$ we have $[\alpha]_z = 1$. In particular this holds for $r(x)$ and all z such that $r(x) < z$. So $[\Box\alpha]_x = 1$. Now we cannot have $[\sim \Box^2\alpha]_x = 1$ since then by (61) we shall have $[\sim \Box\alpha]_{r(x)} = 1$.

AXIOM 56. $[\Box\phi \wedge \Box\psi \wedge \sim \Box^2\psi]_x = 1$ implies $[\sim \Box\psi]_{r(x)} = 1$ and $[\phi]_{r(x)} = 1$ and by (61) $r(x) < y \wedge y \neq r(x)$ implies $[\Box\psi]_{r(x)} = 1$. Conversely $[\Box\psi]_y = 1$ implies $r(x) \neq y$ and $r(x) < y$. Now assume $[\Box\psi]_y = 1$ so $r(x) < y$ and $r(x) \neq y$ and so $r(y) = r(x)$ or $r(x) < r(y)$ and therefore for all z such that $r(y) < z$ or $z = r(y)$ we have $r(x) < z$ or $r(x) = z$ and so $[\phi]_z = 1$. So $[\Box\phi]_y = 1$.

We now want to show that the interpretation (60) is faithful. For this purpose we give a series of lemmas and a Henkin-Scott type completeness proof and construct a model with relations $<$ and r, that fulfill the requirements of the semantics of M_* and such that the modal accessibility relation R of our model is related to $<$ and r exactly according to the syntactical translation of (60).

LEMMA 62. *Let Δ be a complete theory that $\Box\psi \wedge \sim \Box^2\psi \in \Delta$. Then the following set is consistent.*

$$\{\alpha \mid \Box\alpha \in \Delta\} \cup \{\sim \Box\psi\} \cup \{\Diamond\phi \mid \phi \in \Delta\}.$$

PROOF. Otherwise

$$\vdash \alpha_1 \wedge \cdots \wedge \alpha_n \to (\Diamond\phi_1 \wedge \cdots \wedge \Diamond\phi_m \to \Box\psi)$$

also

$$\vdash \Diamond(\phi_1 \wedge \cdots \wedge \phi_m) \to \Diamond\phi_1 \wedge \cdots \wedge \Diamond\phi_m$$

therefore

$$\vdash \bigwedge_i \alpha_i \to (\Diamond \bigwedge_i \phi_i \to \Box\psi), \ \vdash \Box \bigwedge_i \phi_i \to \Box(\Diamond \bigwedge_i \phi_i \to \Box\psi).$$

So

$$\Box(\Diamond \bigwedge \phi_i \to \Box\psi) \in \Delta.$$

Since $\wedge\phi_i \in \Delta$ we get by (51) $\diamond(\diamond \bigwedge_i \phi_i \wedge \sim\Box\psi) \in \Delta$ that is

$$\sim\Box(\sim\diamond \bigwedge_i \phi_i \vee \Box\psi) \in \Delta \text{ or } \sim\Box(\diamond \bigwedge_i \phi_i \rightarrow \Box\psi) \in \Delta,$$

a contradiction.

Extend this set to a complete theory and denote it by Δ_ψ.

LEMMA 65. *In notation of* (62) *the following is consistent*

$$\{\alpha \mid \Box\alpha \in \Delta\} \cup \{\Box^2\psi \wedge \sim\Box^3\psi\} \cup \{\sim\Box\beta \mid \sim\beta \in \Delta\} \cup \{\Box^2\gamma \mid \Box\gamma \in \Delta\}.$$

PROOF. Otherwise

$$\vdash \bigwedge_i \alpha_i \wedge \Box^2\psi \wedge \sim\Box^3\psi \wedge \bigwedge_i \Box^2\gamma_i \rightarrow \bigvee_i \Box\beta_i$$

also

$$\vdash \bigvee_i \Box\beta \rightarrow \Box \bigvee_i \beta_i$$

so

$$\vdash \bigwedge_i \alpha_i \rightarrow (\Box^2\psi \wedge \sim\Box^3\psi \wedge \Box^2 \bigwedge_i \gamma_i \rightarrow \Box \bigvee_i \beta_i).$$

We conclude by necessitation and the fact that $\Box \bigwedge_i \alpha_i \in \Delta$ that

$$\Box(\Box^2\psi \wedge \sim\Box^3\psi \wedge \Box^2 \bigwedge_i \gamma_i \rightarrow \Box \vee \beta_i)$$

is in Δ.

Now by (53) since $\Box \wedge \gamma_i \in \Delta$ and $\sim(\vee \beta_i) \in \Delta$ and $\Box\psi \wedge \sim\Box^2\psi \in \Delta$ we get that

$$\diamond(\diamond \sim (\vee \beta_i) \wedge \Box^2\psi \wedge \sim\Box^3\psi \wedge \Box^2 \wedge \gamma_i)) \in \Delta.$$

And so

$$\sim\Box(\sim\diamond \sim \vee \beta_i \vee \sim(\Box^2\psi \wedge \Box^3\psi \wedge \Box^2 \bigwedge_i \gamma_i)) \in \Delta$$

that is

$$\sim\Box(\Box^2\psi \wedge \sim\Box^3\psi \wedge \Box^2 \wedge \gamma_i \rightarrow \Box \vee \beta_i) \in \Delta$$

a contradiction.

Extend this theory to a complete Δ^s.

DEFINITION 67. $\Delta R \Delta'$ iff (def) $\Box\phi \in \Delta \rightarrow \phi \in \Delta'$.

LEMMA 68. $\Delta_\psi R \Delta$, $\Delta R \Delta_\psi$, $\Delta^s R \Delta$, $\Delta R \Delta^s$.

PROOF. By construction.

LEMMA 69. $\Delta_{\psi_1} = \Delta_{\psi_2}$.

PROOF. Assume $\Box\psi_1 \wedge \sim\Box^2\psi_1 \wedge \Box\psi_2 \wedge \sim\Box^2\psi_2 \in \Delta$, and let $\alpha \in \Delta_{\psi_1}$ and $\sim\alpha \in \Delta_{\psi_2}$.
By (54) we have that

$$\Box(\sim\Box\psi_1 \to \alpha) \in \Delta \text{ or } \Box(\sim\Box\psi_1 \to \sim\alpha) \in \Delta.$$

Now since $\Delta R \Delta_{\psi_1}$ the second cannot be in Δ so $\Box(\sim\Box\psi_1 \to \alpha) \in \Delta$. We now conclude by (56) and (68) that $\Box\psi_1 \leftrightarrow \Box\psi_2 \in \Delta_{\psi_2}$ and $(\sim\Box\psi_1 \to \alpha) \in \Delta_{\psi_2}$ which contradicts $\sim\alpha \in \Delta_{\psi_2}$.

LEMMA 70. $(\Delta^s)_{\Box\psi} = \Delta$.

PROOF. We have to show that

$$\{\alpha \mid \Box\alpha \in \Delta^s\} \cup \{\sim\Box^2\psi\} \cup \{\Diamond\phi \mid \phi \in \Delta^s\} \subseteq \Delta.$$

This follows by construction. So since $\Box^2\psi \wedge \sim\Box^3\psi \in \Delta^s$ we get that $(\Delta^s)_{\Box\psi}$ is defined and unique.

DEFINITION 71. We now define r, and $<$. $r(\Delta)$ equals Δ_ψ if it exists and Δ otherwise (i.e. if for no ψ do we have $\Box\psi \wedge \sim\Box^2\psi \in \Delta$). Define $\Delta < \Delta'$ as follows:

(a) if $\Delta = r(\Gamma)$ and $r(\Gamma) \neq \Gamma$ (i.e. $\Delta = \Gamma_\psi$) and $\Gamma R \Delta'$ and $\Delta' \neq \Delta$ then $\Delta <_a \Delta'$. In particular we get that $\Gamma_\psi <_a \Gamma$ ($\Gamma R \Gamma$ always holds).

(b) if $r(\Gamma) = \Gamma$ then if $\Gamma R \Delta'$ then $\Gamma <_b \Delta'$. In particular $\Gamma < \Gamma$.

(c) Let $<$ be the transitive closure of $<_a \cup <_b$.

REMARK. We have in particular that $r(\Delta) < \Delta$.

LEMMA 72. *If* $\Delta = r(\Gamma)$ *and* $\Delta < \Delta'$ *then* $\Gamma R \Delta'$.

PROOF. Since $<$ was defined as the transitive closure of $<_a \cup <_b$ it is enough to examine a few cases (i.e. proof by induction on the length of the sequence leading from Δ to Δ').

The first two cases deal with length 1.

Case 1. Corresponding to (a) of (71) it is clear that $\Gamma R \Delta'$. Notice that in our case the $<$ sequence is of length 1 and begins with instance of (a) and so $\Delta \neq \Delta'$.

Case 2. Corresponding to (b) of (71) again $\Gamma R \Delta'$ but we may have $\Delta = \Delta'$.

The remaining cases deal with the induction step.

Now assume $\Delta < \Delta_1 < \cdots < \Delta_n < \Delta'$, by induction hypothesis assume that $\Gamma R \Delta_n$ holds and if $\Delta \neq \Gamma$ (i.e. the sequence begins with use of $<_a$) then also $\Delta \neq \Delta_n$.

Consider case 3, for $\Delta <_a \Delta_1$ and $D_n <_a \Delta'$.

Case 3. Assume that $r(\Gamma) = \Delta$ and $\Gamma R \Delta_n$ and $\Delta \neq \Delta_n$ and $\Delta_n \neq \Phi$, and $\Delta_n \neq \Delta'$ and $\Phi R \Delta'$. Also assume that $\Delta = \Gamma_\psi$.

We want to show that $\Gamma R \Delta'$ and $\Delta \neq \Delta'$.

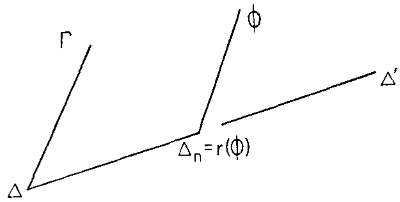


Fig. 3

Let $\Box\beta \in \Gamma$. We must show that $\beta \in \Delta'$.

(a) $\Box^2\beta \in \Gamma$: so $\Box\beta \in \Delta_n$ so we have in Φ the following sentence:

$$\Box\alpha \wedge \sim\Box_\alpha^2 \wedge \Diamond(\Box\beta \wedge \sim\Box\alpha) \quad \text{for some } \alpha$$

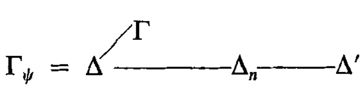
since $\Delta_n = r(\Phi)$ and $\Delta_n \neq \Phi$ holds. So by (55) $\Box^2\beta \in \Phi$ and so $\Box\beta \in \Delta'$ and so $\beta \in \Delta'$.

(b) $\sim\Box^2\beta \in \Gamma$. Therefore $\Gamma_\beta = \Delta$. Now by axiom (56), since $\Delta = \Gamma_\psi$, $\Box(\Box\beta \leftrightarrow \Box\psi) \in \Gamma$. We want to show now that $\Box\psi \in \Delta_n$. Now since $\Delta_n \neq \Delta$, there exists a δ, $\delta \in \Delta \wedge \sim\delta \in \Delta_n$ and so by axiom (54) $\Box(\sim\Box\psi \to \delta) \in \Gamma$ and so $\sim\Box\psi \notin \Delta_n$ since $\sim\delta \in \Delta_n$. We conclude that $\Box\psi \in \Delta_n$ and therefore $\Box\beta \in \Delta_n$. Now continue as in (a).

Now we have that $\Box\psi \in \Delta_n$ (see (b) above) therefore $\Box^2\psi \in \Phi$ by (b) above and so $\Box\psi \in \Delta'$ and therefore $\Delta \neq \Delta'$ (which is the second thing we had to prove).

Consider case 4 for $\Delta <_a \Delta_1$ and $\Delta_n <_b \Delta'$.

Case 4. Assume that $r(\Gamma) = \Gamma_\psi = \Delta$ and $\Gamma R \Delta_n$ and $\Delta \neq \Delta_n$ and $\Delta_n R \Delta'$ and $r(\Delta_n) = \Delta_n$



Assume $\Box\beta \in \Gamma$, we must show that $\beta \in \Delta'$.

In this case again we have $\Box\psi \in \Delta_n$ and so $\Box\beta \in \Gamma \to \Box\beta \in \Delta_n$(by(a) and (b) of case 3).

To show that $\Delta \neq \Delta'$, notice that since $r(\Delta_n) = \Delta_n$, $\Box\psi \in \Delta_n \to \Box^2\psi \in \Delta_n$ so $\Box\psi \in \Delta'$ that is again $\Delta \neq \Delta'$ since $\sim\Box\psi \in \Delta$.

Consider case 5 where $\Delta <_b \Delta_1$ and $\Delta_n <_a \Delta'$.

Case 5. Assume $r(\Delta) = \Delta$, $r(\Phi) = \Delta_n = \Phi_\alpha$, and $\Phi R \Delta'$ and by induction hypothesis $\Delta R \Delta_n$.

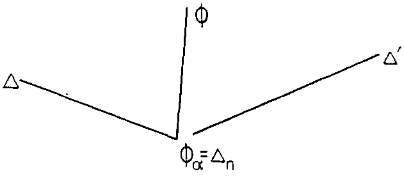


Fig. 4

(e.g. the analogue of Case 3); we must show $\Delta R \Delta'$. But in our case, $r(\Delta) = \Delta$ so $\Box\beta \in \Delta \to \Box^2\beta \in \Delta$ and so we have that (a) of case 3 holds. Consider case 6 with $\Delta <_b \Delta_1$ and $\Delta_n <_b \Delta'$.

Case 6. Let $r(\Delta) = \Delta$, $\Delta R \Delta_n$, $r(\Delta_n) = \Delta_n$.

In this case $\Box\beta \in \Delta \to \Box^2\beta \in \Delta$ and so $\Box\beta \in \Delta_n$ and so $\Box^2\beta \in \Delta_n$ and so $\Box\beta \in \Delta$.

Thus Lemma (72) is proved.

THEOREM 73. *$\Delta < \Gamma$ and $\Delta \neq r(\Gamma) \to \Delta < r(\Gamma)$.*

PROOF. If $\Delta < \Gamma$ we have a sequence leading from Δ to Γ.

Case 1. The sequence begins with $r(\Delta) = \Delta$, (Case b).

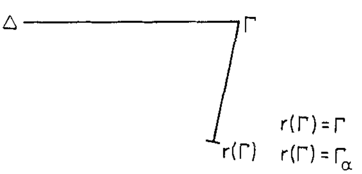


Fig. 5

Let us show that $\Delta R r(\Gamma)$. Let $\Box\beta \in \Delta$. We have $\Box\beta \in \Delta \to \Box^2\beta \in \Delta \to \Box\beta \in \Gamma \to \beta \in r(\Gamma)$. Now we have $\Delta R\, r(\Gamma)$ and so $\Delta < r(\Gamma)$, by definition.

Case 2. The sequence begins with case (a), i.e., $\Delta = r(\Phi)$, $\Delta \neq \Phi$. So by what we proved in (72) (more specifically the induction hypothesis of (72))

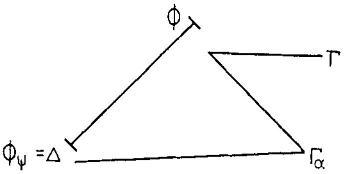


Fig. 6

We get $\Phi R\Gamma$ and $\Delta \neq \Gamma$.

Now assume $\Box\beta \in \Delta$ and $\sim\beta \in \Gamma_\alpha$ and so $\sim\Box\beta \in \Gamma$. Now since $\Phi R\Gamma$ we have $\sim\Box^2\beta \in \Phi$ and therefore $\Phi_\beta = \Phi_\psi$ i.e. $\sim\Box\beta \in \Delta$. Now $\Delta \neq \Gamma$, so for some δ, $\delta \in \Delta$, $\sim\delta \varepsilon \Gamma$. We then have, by (54) $\Box(\sim\Box\beta \to \delta) \in \Phi$, so $\sim\Box\beta \to \delta \in \Gamma$ so $\delta \in \Gamma$, a contradiction.

So we have that $\Delta R\Gamma_\alpha$ and since $\Gamma_\alpha \neq \Delta$ we get that $\Delta < \Gamma_\alpha$.

THEOREM 74. *$\Delta R\Delta'$ iff ($r(\Delta) = \Delta'$ or $r(\Delta) < \Delta'$).*

PROOF. If $\Delta R\Delta'$ and $r(\Delta) \neq \Delta'$ then $r(\Delta) < \Delta'$ by definition, if $r(\Delta) = \Delta'$ then $\Delta R\Delta'$. If $r(\Delta) < \Delta'$ then $\Delta R\Delta'$ by (72).

THEOREM 75. *$<$ and r fulfill Theorem 75. $<$ and r fulfill:*(11)–(14).

PROOF.

(10) by definition

(11) by (71)(a) and (b)

(12–13) by (70) and the definition of r.

(14) by (73).

THEOREM 76. *G_* is exactly the set of all sentences with $\Box$, $\wedge$ and $\sim$ that hold in M_*.*

PROOF. One direction is (60). Assume θ to be a complete G_* theory. Construct a Makinson-Scott model of Θ, i.e. T is the set of all complete and consistent G_* theories, $\Delta R\Delta'$ iff $\Box\phi \in \Delta \to \phi \in \Delta'$ and for atomic ϕ $[\phi]_\Delta = 1$ iff $\phi \in \Delta$, the result then holds for any ϕ.

Define $<$, r on T. By (75) we get an M^* structure. By (74) we still have $[\phi]_\Delta$ in $M_* = 1$ iff $\phi \in \Delta$.

THEOREM 77. *G_* is an extension of T which is decidable, finitely axiomatizable, normal and lacks the final model property.*

Note added in proof. Kit Fine has found extensions of $S4$ and of the intuitionistic propositional calculus that lack the f.m.p. The present method can be used to obtain extensions of K. Fine's systems that are decidable and lack the f.m.p.

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